Pseudoconformal field theory at "wrong level"

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Abstract

Pseudoaffine theories are characterized by formal replacement of the level by the fractional number: $k \to \frac{k}{q}$, where q is an integer co-prime with $k(g+k)$ (g being dual Coxeter number). An example of "forbidden" q is considered (SU(2), $q=2$). The generalized pseudoaffine theory is obtained. Its fusions are similar to the affine ones, number of fields in the spectrum is the integer multiple of the number in the affine case, central charge is the integer multiple of the affine one. Spectra of minimal models are calculated.

1 Introduction

Classification of all the CFT's with the given fusion rules is a difficult problem. Verlinde formula provides us with a hint [1], given the fusions:

$$N_{ij}^{k} = \sum_{m} \frac{S_{im} S_{jm} S^{\dagger}_{mk}}{S_{0m}} \tag{1}$$

first find all the possible S matrices, then try to realize them as the full fledged CFT's.

The simplest case is boson on a lattice [2]. For λ, μ vectors of the lattice M the S-matrix is:

$$S_{\lambda,\mu} = |M^{*}/M|^{-\frac{1}{2}} \exp(-2\pi i < \lambda, \mu >) \tag{2}$$

If M^*/M is a cyclic group $Z_{M^*/M}$ then the general solution of Verlinde equation is:

$$\begin{array}{c} S_{\lambda,\mu} = |M^*/M|^{-\frac{1}{2}} \exp(-2\pi i q < \lambda, \mu >) \\ (q, k|M^*/M|) = 1 \end{array} \tag{3}$$

(q is coprime with $|M^*/M|$), it is just an automorphism of a cyclic group. The dimensions of fields in the corresponding "pseudobosonic" theory is[1]: $h_\lambda = \frac{q\lambda^2}{2k}$. In general M^*/M is a sum of cyclic groups so one can take any automorphism of it:

$$S_{\lambda,\mu} = |M^*/M|^{-\frac{1}{2}} \exp(-2\pi i < h(\lambda), \mu >) \tag{4}$$

As was found in ([2],[4]) the pseudobosonic theories really exist, i.e. for any such automorphism one can find another lattice $\tilde{M}$ such that (4) is its S matrix.

One might ask: what if the condition on q in (3) is violated? Is there any theory that has S matrix "close to the (3)"? The natural generalization of the pseudobosonic theory is boson at level $q^{2n-1} \times k,\ n \in\ Z_+$. Among the states in the bosonic spectrum there are states of dimension:

$$\Delta^{(q)}_{q^n m} = \frac{(q^n m)^2}{4q^{2n-1}k} = q\frac{m^2}{4k} = q\Delta^{(1)}_m$$

These are states that one naively would propose for the forbidden value of q. Their modular properties are as in (3) so that the S-matrix of (3) is a part of the bigger S-matrix of the "generalized pseudobosonic theory". n is a positive integer, so there are many theories with the S-matrix that has (3) as a submatrix. Ths phenomenon will also occur later.

Similarly one can consider the affine Lie algebras and try to classify the theories that have the same fusion rules. The general solution of the Verlinde formula is [1]:

$$\begin{array}{c} S_{\Lambda,\Lambda'} = i^{|\Delta|} \left|\frac{M^*}{(k+g)M}\right|^{-\frac{1}{2}} \sum_{w\in W} (-1)^w \exp(-2\pi i q \frac{<w(\Lambda+\rho),(\Lambda'+\rho)>}{(k+g)}) \\ (q, g(k+g)) = 1 \end{array} \tag{5}$$

Such theories will be called "q pseudoaffine theories" or just pseudoaffine, the $q = 1$ case is the usual affine theory. The q pseudoaffine theory has central

[1] Here k is the "level" of the U(1) boson on the lattice M, i.e. just $|M^*/M|$

charge:

$$c(q) = qc(1) = \frac{qk\mathrm{dimG}}{k+g}\mathrm{mod}\ 4$$

and spectrum with dimensions:

$$h_\lambda^{(q)} = qh_\lambda^{(1)} = \frac{q < \lambda, \lambda + 2\rho >}{2(k+g)}\mathrm{mod}\ \mathrm{Z}$$

To realize these theories ([2]) one can use decomposition of Hilbert space into a product of parafermions and free bosons ([5],[7],[8]) (decomposition of characters into string functions:

$$\chi^\Lambda(\tau) = \sum_\lambda C_\lambda^\Lambda(\tau)\theta_\lambda(\tau) \tag{6}$$

where $C_\lambda^\Lambda(\tau)$ are parafermionic characters). (There exist many different parafermionic theories, the parafermions that appear here will be called $q = 1$ parafermions). By changing the bosonic lattice to another one with the same fusions different "pseudoaffine" theories are constructed. So e.g. for $q = p(k+g)+1$, p is integer, we have: $c(q)-c(1) =$ integer and the corresponding theories can be realized as a product of parafermions and pseudo-bosons with $\tilde{q} = \tilde{p}k + 1$ and $\tilde{p}$ satisfies: $\frac{p<\Lambda,\Lambda+2\rho>}{2} = \frac{\tilde{p}\lambda^2}{2}$ mod Z. However in this way not all the theories are obtained.

To realize more theories consider the following construction:

- Given the parafermions from (6) (the $q = 1$ parafermions), with the spectrum ϕ_α, conformal dimensions h_α, central charge $c = c(1)$ and fusion rules N_{ij}^k try to find for some integer q the new theory with the central charge $c(q) = qc(1)$, spectrum ϕ_α with conformal dimensions $q * h_\alpha$ and the same fusion rules N_{ij}^k. If such a model exist it will be called q parafermions.

Note that the idea of procedure is the same as the initial question: find the theory with the given fusions. Therefore we go once more to the modular matrix. The parafermions of (6) are obtained from the coset $\frac{G}{U(1)^{\mathrm{rank(G)}}}$ thus the parafermionic modular matrix is:

$$S_{\mathrm{parafermions}}^{(\Delta,\lambda),(\Delta'\lambda')} = S_{\mathrm{affine}}^{\Delta,\Delta'}(\bar{S}_{\mathrm{boson}})^{(\lambda,\lambda')} \tag{7}$$

Therefore the q parafermions exist iff there exist the corresponding pseudoaffine and pseudobosonic theories. As follows from (3) and (5) this occurs when $(q, g(k+g)) = 1$ and $(q, k|M^*/M|) = 1$. Finally:

$$(q, gk(k+g)) = 1 \tag{8}$$

In what follows this formal definition of q parafermions will be used in the cases when (8) is violated, just to compare to the existing models with similar properties.

Having found the q-parafermions for some q one can again play with bosons, i.e. to multiply the parafermions by bosons on different lattices but with the same fusions. In such a way the full hierarchy of pseudoaffine theories is obtained.

A natural question appears: what happens when the condition (8) is violated? In this case (by analogy with the generalized pseudobosons) one can try to find a theory that is a generalization of q-parafermions in the following sense

- its central charge is the same as the central charge of q-parafermions
- its spectrum consist of two parts: the "old" fields (that appear in the q-parafermionic spectrum) and the "new" fields (that do not appear there)
- When "neglecting" the new fields, fusions of the old fields are the same as of q-parafermions:
$$\hat{N}_{ij}^k = N_{ij}^k$$

Certainly there exist a lot of such theories. A general way to realize some general pseudo CFT at level $\frac{k}{q}$ is to take the the q times product of original CFT. In particular here one can take a multiple tensor product of the $q = 1$ parafermions with themselves or orbifolds of this product etc. We are interested, however, in the "smallest deformation" of the q-parafermions, i.e. the number of "new" fields in the spectrum should be as small as possible.

In this paper a particular case is considered: for $\frac{SU(2)_N}{U(1)}$ parafermions (the Zamolodchikov-Fateev parafermions) the $q = 2$ is taken. It happens that for each N there exist a model of the central charge $c(2) = 2c(1)$. Its spectrum includes that of $q = 2$ parafermions ("old fields") and contains additional

fields ("new fields"). Fusions of "old fields" are almost the same as for $q = 2$ parafermions (in the sense of previous paragraph):

$$N_{ij}^{k} = \hat{N}_{ij}^{k}$$

The next step is to couple the "generalized pseudo-parafermions" to "generalized pseudo-bosons" previously described. We take the boson at level $q^{2n-1}k,\ n \in\ Z_+$. Then among the states in the bosonic spectrum there are states of dimension:

$$\Delta_{q^n m}^{(q)} = \frac{(q^n m)^2}{4q^{2n-1}k} = q\frac{m^2}{4k} = q\Delta_m^{(1)}$$

These corresponds to the "old fields" in the parafermionic part and are coupled to them. Other fields in the bosonic spectrum are coupled to the "new fields" in the parafermionic part, so that the "generalized pseudoaffine theory" is constructed. This theory has the following properties:

- Its central charge is related to that of the initial affine theory:

$$c(q) = qc(1) \mathrm{mod}\ 4$$

- Its spectrum includes that of the q pseudoaffine theory. Number of states in the spectrum is an integer multiple of that in the initial affine theory.

- Denoting fusions in the affine theory by $N_{\mu\nu}^{\lambda}$ and in the generalized pseudoaffine by $\hat{N}_{\mu\nu}^{\lambda}$ one has:

$$N_{\mu\nu}^{\lambda} = \hat{N}_{\mu\nu}^{\lambda}$$

2 Parafermions

2.1 General background

Z_N parafermions are defined as a collection of fields $(\psi_i)_{i=1}^{N-1},\ \psi_i \equiv \psi_{N-i}^{\dagger}$ with fusions: $[\psi_i] \times [\psi_j] = [\psi_{(i+j)\mathrm{modN}}]$, of conformal dimensions satisfying:

$\Delta_k = \Delta_{N-k}$ with OPE's:

$$\begin{array}{c} \psi_i(z)\psi_j(w) \sim \frac{C_{i,j}}{(z-w)^{\Delta_i+\Delta_j-\Delta_{i+j}}} \left(\psi_{i+j}(w) + (z-w)\frac{\Delta_{i+j}+\Delta_i-\Delta_j}{2\Delta_{i+j}}\partial\psi_{i+j}(w) + \dots\right), \ i+j \neq N \\ \psi_i(z)\psi_i^{\dagger}(w) \sim \frac{1}{(z-w)^{2\Delta_i}} \left(1 + (z-w)^2\frac{2\Delta_i}{c}T(w) + \dots\right) \end{array} \tag{9}$$

here by $i+j$ it is meant $i+j$ mod N, T is the energy-momentum tensor of the parafermions, $C_{i,j}$ are the structure constants that satisfy several constraints from Jacobi identities.

As was noticed in [5] the monodromy condition for parafermions allows conformal dimensions of the general form: $\Delta_i = q\frac{i(N-i)}{N} + m_i, m_i = m_{N-i} \in Z$. The q parafermions are obtained when[2] $m_i = 0$

The Zamolodchikov-Fateev parafermions [5] are the $q = 1$ parafermions: $\Delta_k = \frac{k(N-k)}{N}$. In this case the central charge is fixed: $c = \frac{2(N-1)}{N+2}$. The parafermions are described by $\frac{SU(2)_N}{U(1)}$ ([5],[7]). Fields in spectrum are: $\chi^l_m, l = 0, \dots, N,\ m \in (-N+1, N),\ m = l$ mod 2. The fractional part of their conformal dimensions are

$$h = \frac{l(l+2)}{4(N+2)} - \frac{m^2}{4N} \tag{10}$$

The Z_N parafermions with $\Delta_k = 2\frac{k(N-k)}{N}$ were explored partially in [6]. As noted in the introduction for q=2 the fusions of the spectrum are necessarily different from the ones of the q=1 case. Our goal is to make them as similar to the q=1 case as possible. In particular for N even there is a parafermionic field: $\chi^0_N \equiv \psi_{N/2}$ with the very specific fusions:

$$[\psi_{N/2}] \times [\psi_{N/2}] = [\psi_N] \equiv [0] \quad [\psi_{N/2}] \times [\psi_i] = [\psi_{N/2+i}],\ [\psi_{N/2}] \times [\psi_{N/2+i}] = [\psi_i] \tag{11}$$

It has no analogs in the q=1 case.Since this field is of integer dimension $\frac{N}{2}$ one can extend the chiral operator algebra. Fusions of this field with other fields:

$$[\chi^0_N] \times [\chi^\lambda_\nu] = [\chi^\lambda_{\nu+N}] \tag{12}$$

show that it is local[3], and all the fields of the coset are local with respect to it, so modular invariance does not forbid any representation. Therefore

[2] Several examples with nonzero m_i were explored in [11], however no new unitary models were found.

[3] A field is local with respect to another if their OPE contains only integer powers of $(z-w)$. A field is local if it is local with respect to itself.

the representations of such "extended" algebra are denoted by $\chi_m^l,\ 0 \leq l \leq N,\ 0 \leq m \leq N$ for general even N. We will consider the q=2 parafermions for odd N and q=2 extended algebra for even N.

The coset corresponding to the q=2 parafermions is[4] ([9]): $\frac{SU(N)_1 \oplus SU(N)_1}{SO(N)_4^D}$ for $N \geq 4$ and $\frac{SU(3)_1 \oplus SU(3)_1}{SO(3)_8^D}$ for $N = 3$. In the original paper ([9]) more general models $\frac{SO(N)_k \oplus SU(N)_1}{SO(N)_{2+k}}$ were considered. Their central charge:

$$c = (N-1)(1 - \frac{N(N-2)}{(k+N)(k+N-2)}) \tag{13}$$

For $k = 2$ it coincides with the central charge of q=2 parafermions. In this case (due to conformal embedding: $\widehat{SO(N)}_2 \subset \widehat{SU(N)}_1$) the chiral algebra can be extended (it corresponds to the D-modular invariant of the coset) and we extend it to obtain the spectrum similar to that of q-parafermions. For $N = 3$, due to specialities of $SO(N)$, the coset is: $\frac{SU(2)_4 \oplus SU(3)_1}{SU(2)_8}$ (conformal embedding: $\widehat{SU(2)}_4 \subset \widehat{SU(3)}_1$). The cases of low N (N=3,4,6) are dealt with separately due to relations: $SO(3) \approx SU(2)$, $SO(4) \approx SU(2) \oplus SU(2)$, $SO(6) \approx SU(4)$.

2.2 Technicalities

To obtain the selection rules and field identifications of the coset G/H the projection matrix is used (see [10] for a concise introduction). This is the $r_h \times r_g$ matrix that projects fundamental weights of algebra G to the fundamental weights of its subalgebra H. As a simple example consider the embedding: $SU(2) \subset SU(3)$. Take first the module [1,0] of $SU(3)$. This module can be decomposed into irreducible modules of $SU(2)$: [0], [1], [2].

The projection matrix for the decompostition $[1,0] \rightarrow [2]$ is

$$P = (2, 2)$$

. The relation between the generators of $SU(3)$ and $SU(2)$ is (in the Cartan-Weyl basis):

$$J^{\pm} = 2(E^{\pm\alpha_1} + E^{\pm\alpha_2})$$
$$J^0 = 2(H^1 + H^2)$$

[4] As was noted in the introduction many cosets have the same central charge: $[\frac{SU(N)_1 \oplus SU(N)_1}{SU(N)_2}]^2$, $\frac{SU(N)_1 \oplus SU(N)_1}{SU(N)_2} \times \frac{SU(N)_2}{SO(N)_4}$ etc. We need however the spectrum that contains spectrum of q=2 parafermions and fusions similar to the $q = 1$ case.

The case of decomposition: $[1,0] \to [1] \oplus [0]$ is realized by $P = (1,1)$. The realization in terms of generators is:

$$J^{\pm} = \sqrt{2} E^{\pm(\alpha_1+\alpha_2)}$$
$$J^0 = H^1 + H^2$$

Having found the projection matrix one can decompose all the $SU(3)$ modules with respect to $SU(2)$, e.g. for $P = (2,2)$:

$$[0,1] \to [2] \quad [1,1] \to [4] \oplus [2]$$

To the same decompositions of G modules with respect to H there generally correspond many projection matrices. More coarser object is index of embedding of $H \subset G$:

$$x_e = \frac{|P\theta_G|^2}{|\theta_H|^2}, \quad (\theta_G,\ \theta_H \text{ are the highest roots of G and H}) \tag{14}$$

It is unique for a given decomposition, however different decompositions can have the same x_e. If $\hat{H}_{k_h} \subset \hat{G}_{k_G}$ and the index of embedding is x_e then

$$k_H = x_e \times k_G$$

The selection rules for the field χ^{λ}_{ν} of G/H are:

$$P * \lambda - \nu \in P * Q, \quad Q \text{ is the root lattice of G} \tag{15}$$

Field identification takes place when the nontrivial branching $A \to \tilde{A}$ of outer automorphism of Dynkin diagram occurs:

$$\forall \lambda \in G: \ (A\hat{\omega}_0, \lambda) = (\tilde{A}\hat{\omega}_0, P * \lambda) \ \ \mathrm{modZ} \tag{16}$$

In this case the fields χ^{λ}_{ν} and $\chi^{A\lambda}_{\tilde{A}\nu}$ are identified.
As an example consider the $SU(2) \subset SU(3)$ embedding. The outer automorphisms of SU(3) ($A\hat{\omega}_0 = \hat{\omega}_1, \quad A^2\hat{\omega}_0 = \hat{\omega}_2$) has no nontrivial branching since:

$$(A\hat{\omega}_0, \lambda) = \frac{2\lambda_1 + \lambda_2}{3}, \quad (A^2\hat{\omega}_0, \lambda) = \frac{\lambda_1 + 2\lambda_2}{3}$$

The SU(2) automorphism: ($\tilde{A}\hat{\omega}_0 = \hat{\omega}_1$) branches nontrivially:
$(\tilde{A}\hat{\omega}_0, P\lambda) = \lambda_1 + \lambda_2 = 0 \text{ mod Z} = (1\hat{\omega}_0, \lambda)$.
So in this case the fields χ^{λ}_{ν} and $\chi^{\lambda}_{\tilde{A}\nu}$ are identified.

2.3 $\frac{SU(3)_1 \oplus SU(3)_1}{SU(2)_8^D}$

The embedding $\widehat{SU(2)}_4 \subset \widehat{SU(3)}_1$ has index $x_e = 4$, the projection matrix is: $P = (2,2)$. Therefore the selection rule for the field[5] χ^{λ,μ_ν} is: $\nu_1 = 0 \bmod 2$ There is a nontrivial branching of Dynkin diagram outer automorphism: $A * \hat{\omega}_0 = \hat{\omega}_1$. Corresponding to it the field identification is $\chi^{\lambda,\mu}_{\nu_0,\nu_1} \equiv \chi^{\lambda,\mu}_{\nu_1,\nu_0}$. The spectrum of the theory is summarized in the table. The spectrum of the q=2 parafermions is given for comparison.

Spectrum of the q=2 Z_3 parafermions

χ^0_{-2}	$\frac{4}{3}$	χ^0_0	0	χ^0_2	$\frac{4}{3}$
χ^1_{-1}	$\frac{2}{15}$	χ^1_1	$\frac{2}{15}$	χ^1_3	$\frac{4}{5}$

Spectrum of the $\frac{SU(3)_1 \oplus SU(3)_1}{SU(2)_8}$

$\chi_0^{0,0}$	0	$\chi_2^{0,0}$	$\frac{4}{5}$	$\chi_4^{0,0}$	$\frac{2}{5}$
$\chi_0^{1,0}$	$\frac{4}{3}$	$\chi_2^{1,0}$	$\frac{2}{15}$	$\chi_4^{1,0}$	$\frac{11}{15}$
$\chi_0^{2,0}$	$\frac{4}{3}$	$\chi_2^{2,0}$	$\frac{2}{15}$	$\chi_4^{2,0}$	$\frac{11}{15}$
$\chi_0^{0,1}$	$\frac{4}{3}$	$\chi_2^{0,1}$	$\frac{2}{15}$	$\chi_4^{0,1}$	$\frac{11}{15}$
$\chi_0^{0,2}$	$\frac{4}{3}$	$\chi_2^{0,2}$	$\frac{2}{15}$	$\chi_4^{0,2}$	$\frac{11}{15}$
$\chi_0^{1,1}$	$\frac{2}{3}$	$\chi_2^{1,1}$	$\frac{7}{15}$	$\chi_4^{1,1}$	$\frac{1}{15}$
$\chi_0^{2,1}$	$\frac{2}{3}$	$\chi_2^{2,1}$	$\frac{7}{15}$	$\chi_4^{2,1}$	$\frac{1}{15}$
$\chi_0^{1,2}$	$\frac{2}{3}$	$\chi_2^{1,2}$	$\frac{7}{15}$	$\chi_4^{1,2}$	$\frac{1}{15}$
$\chi_0^{2,2}$	$\frac{2}{3}$	$\chi_2^{2,2}$	$\frac{7}{15}$	$\chi_4^{2,2}$	$\frac{1}{15}$

As one can see the two spectra are similar, the difference is in the additional fields of the coset. The fusions are "almost the same" (here and in the sequel we give only part of the fusion rules to illustrate the situation):

Fusion of the q=2 Z_3 parafermions

$[\chi^0_2] * [\chi^0_{-2}] = [\chi^0_0]$
$[\chi^0_2] * [\chi^0_2] = [\chi^0_{-2}]$
$[\chi^1_3] * [\chi^1_3] = [\chi^0_0] + [\chi^1_3]$
$[\chi^1_1] * [\chi^1_1] = [\chi^0_{-2}] + [\chi^1_{-1}]$

Fusion of the $\frac{SU(3)_1 \oplus SU(3)_1}{SU(2)_8^D}$

$[\chi_0^{1,0}] * [\chi_0^{2,0}] = [\chi_0^{0,0}]$
$[\chi_0^{1,0}] * [\chi_0^{1,0}] = [\chi_0^{2,0}]$
$[\chi_2^{0,0}] * [\chi_2^{0,0}] = [\chi_0^{0,0}] + [\chi_2^{0,0}] + [\chi^{0,0,4}]$
$[\chi_2^{1,0}] * [\chi_2^{1,0}] = [\chi_0^{2,0}] + [\chi_2^{2,0}] + [\chi_4^{2,0}]$

[5]Here and in the sequel the weights of algebras are given in the basis of fundamental weights: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) = \lambda_1 * \hat{\omega}_1 + \lambda_2 * \hat{\omega}_2 + \ldots + \lambda_r * \hat{\omega}_r$. The "imaginary" weight $\lambda_0 * \hat{\omega}_0$ of affine Lie algebras is omitted for brevity. In the case of $\widehat{SU(N)}_1$ the basic representations: $(\underbrace{0, \ldots, 0, 1}_{m}, 0, \ldots, 0) = \hat{\omega}_m, m \geq 0$ are denoted by (m)

2.4 $\frac{SU(4)_1 \oplus SU(4)_1}{SU(2)_4 \oplus SU(2)_4}$

The projection matrix for each of the SU(4) factors is: $P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.
The selection rules for a field $\chi^{\lambda,\mu}_{\nu^{(1)}\nu^{(2)}}$ ($\nu^{(1)}$ and $\nu^{(2)}$ corresponding to the two SU(2) factors) are:
$\lambda_1 + \lambda_3 + \mu_1 + \mu_3 - \nu_1^{(1)} = \lambda_1 + \lambda_3 + \mu_1 + \mu_3 - \nu_1^{(2)} = 0 \bmod 2$. Field identification arises from the nontrivial branching of SU(4) automorphism: $A\hat{\omega}_0 = \hat{\omega}_2$ into the SU(2) one: $\tilde{A}\hat{\omega}_0 = \hat{\omega}_1$. It gives: $\nu_0^{(1)} \geq \nu_1^{(1)}, \nu_0^{(2)} \geq \nu_1^{(2)}$. The two spectra are given in the table:

Spectrum of the q = 2 Z_4 extended parafermions

χ_0^0	0	χ_2^0	$\frac{3}{2}$
χ_1^1	$\frac{1}{8}$	χ_3^1	$\frac{9}{8}$
χ_0^2	$\frac{2}{3}$	χ_2^2	$\frac{1}{6}$

Fractional parts of dimensions of the $\frac{\mathrm{SU(4)_1 \oplus SU(4)_1}}{\mathrm{SO(4)_4}}$

$\chi_0^{0,0}$	0	$\chi_{0,2}^{0,0}$	$\frac{2}{3}$	$\chi_{2,0}^{0,0}$	$\frac{2}{3}$	$\chi_{2,2}^{0,0}$	$\frac{1}{3}$
$\chi_{1,1}^{0,1}$	$\frac{1}{8}$	$\chi_{1,1}^{1,0}$	$\frac{1}{8}$				
$\chi_0^{0,2}$	$\frac{1}{2}$	$\chi_{0,2}^{0,2}$	$\frac{1}{6}$	$\chi_{2,0}^{0,2}$	$\frac{1}{6}$	$\chi_{2,2}^{0,2}$	$\frac{5}{6}$
$\chi_0^{2,0}$	$\frac{1}{2}$	$\chi_{0,2}^{2,0}$	$\frac{1}{6}$	$\chi_{2,0}^{2,0}$	$\frac{1}{6}$	$\chi_{2,2}^{2,0}$	$\frac{5}{6}$
$\chi_0^{1,1}$	$\frac{3}{4}$	$\chi_{0,2}^{1,1}$	$\frac{5}{12}$	$\chi_{2,0}^{1,1}$	$\frac{5}{12}$	$\chi_{2,2}^{1,1}$	$\frac{1}{12}$
$\chi_{1,1}^{1,2}$	$\frac{5}{8}$	$\chi_{1,1}^{2,1}$	$\frac{5}{8}$				
$\chi_0^{2,2}$	0	$\chi_{0,2}^{2,2}$	$\frac{2}{3}$	$\chi_{2,0}^{2,2}$	$\frac{2}{3}$	$\chi_{2,2}^{2,2}$	$\frac{1}{3}$
$\chi_{1,1}^{0,3}$	$\frac{1}{8}$	$\chi_{1,1}^{3,0}$	$\frac{1}{8}$				
$\chi_{1,1}^{3,2}$	$\frac{5}{8}$	$\chi_{1,1}^{2,3}$	$\frac{5}{8}$				

Some of the fusion rules:

Fusions of the q = 2 Z_4 extended parafermions

$[\chi_2^0] * [\chi_2^0] = [\chi_0^0]$
$[\chi_2^0] * [\chi_1^1] = [\chi_3^1]$
$[\chi_2^0] * [\chi_4^2] = [\chi_2^2]$
$[\chi_1^1] * [\chi_1^1] = [\chi_2^0] + [\chi_2^2]$

Fusions of of $\frac{\mathrm{SU(4)_1 \oplus SU(4)_1}}{\mathrm{SO(4)_4}}$

$[\chi_0^{0,2}] * [\chi_0^{0,2}] = [\chi_0^{0,0}]$
$[\chi_0^{0,2}] * [\chi_0^{0,1}] = [\chi_0^{0,3}]$
$[\chi_0^{0,2}] * [\chi_2^{0,0}] = [\chi_2^{0,2}]$
$[\chi_{1,1}^{0,1}] * [\chi_{1,1}^{0,1}] = [\chi_{0,0}^{0,2}] + [\chi_{2,0}^{0,2}] + [\chi_{0,2}^{0,2}] + [\chi_{2,2}^{0,2}]$

2.5 $\frac{SU(6)_1 \oplus SU(6)_1}{SU(4)_4}$

The projection matrix for each SU(6) factor is: $P = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$. The selection rules for a field $\chi_\nu^{\lambda,\mu}$:

$$\begin{pmatrix} \lambda_2 + \lambda_4 + \mu_2 + \mu_4 - \nu_1 \\ \lambda_1 + \lambda_5 + \mu_1 + \mu_5 - \nu_2 \\ \lambda_2 + 2\lambda_3 + \lambda_4 + \mu_2 + 2\mu_3 + \mu_4 - \nu_3 \end{pmatrix} = \mathrm{Span}_{\mathrm{Z}} \left(\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right) \tag{17}$$

Field identification arises from the nontrivial branching of each of the SU(6) automorphism: $A\hat{\omega}_0 = \hat{\omega}_3$ into the SU(4) one: $\tilde{A}\hat{\omega}_0 = \hat{\omega}_1$. It gives: $\nu_0 \geq \max(\nu_1, \nu_2, \nu_3)$. The two spectra are given in the table[6]:

Spectrum of the q = 2 Z_6 extended parafermions

χ_0^0	0	χ_2^0	$\frac{5}{3}$	χ_4^0	$\frac{8}{3}$
χ_1^1	$\frac{5}{48}$	χ_3^1	$\frac{23}{16}$	χ_5^1	$\frac{101}{48}$
χ_0^2	$\frac{1}{2}$	χ_2^2	$\frac{1}{6}$	χ_4^2	$\frac{7}{6}$
χ_1^3	$\frac{41}{48}$	χ_3^3	$\frac{3}{16}$	χ_5^3	$\frac{41}{48}$

Fractional parts of dimensions of the $\frac{\mathrm{SU(6)_1 \oplus SU(6)_1}}{\mathrm{SU(4)_4}}$

$\chi_{0,0,0}^{0,0}$	0	$\chi_{0,2,0}^{0,0}$	$\frac{1}{4}$	$\chi_{1,0,1}^{0,0}$	$\frac{1}{2}$
$\chi_{0,0,2}^{0,1}$	$\frac{41}{48}$	$\chi_{0,1,0}^{0,1}$	$\frac{5}{48}$	$\chi_{1,1,1}^{0,1}$	$\frac{23}{48}$
$\chi_{0,0,0}^{0,2}$	$\frac{2}{3}$	$\chi_{0,2,0}^{0,2}$	$\frac{11}{12}$	$\chi_{1,0,1}^{0,2}$	$\frac{1}{6}$
$\chi_{0,0,2}^{0,3}$	$\frac{3}{16}$	$\chi_{0,1,0}^{0,3}$	$\frac{7}{16}$	$\chi_{1,1,1}^{0,3}$	$\frac{13}{16}$
$\chi_{0,0,0}^{1,1}$	$\frac{5}{6}$	$\chi_{0,2,0}^{1,1}$	$\frac{1}{12}$	$\chi_{1,0,1}^{1,1}$	$\frac{1}{3}$
$\chi_{0,0,2}^{1,2}$	$\frac{25}{48}$	$\chi_{0,1,0}^{1,2}$	$\frac{37}{48}$	$\chi_{1,1,1}^{1,2}$	$\frac{7}{48}$
$\chi_{0,0,0}^{2,2}$	$\frac{1}{3}$	$\chi_{0,2,0}^{2,2}$	$\frac{7}{12}$	$\chi_{1,0,1}^{2,2}$	$\frac{5}{6}$
$\chi_{0,0,2}^{2,3}$	$\frac{41}{48}$	$\chi_{0,1,0}^{2,3}$	$\frac{5}{48}$	$\chi_{1,1,1}^{2,3}$	$\frac{23}{48}$
$\chi_{0,0,0}^{3,3}$	$\frac{1}{2}$	$\chi_{0,2,0}^{3,3}$	$\frac{3}{4}$	$\chi_{1,0,1}^{3,3}$	0

2.6 $\frac{SU(N)_1 \oplus SU(N)_1}{SO(N)_4}$, $N \geq 5$, odd

The projection matrix for each SU(N) factor is

$$P = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & \dots & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 2 & 2 & 0 & \dots & 0 \end{pmatrix} \tag{18}$$

[6]Part of the coset fields is omitted, e.g. fields $\chi_{0,0,0}^{0,2}$ and $\chi_{0,0,0}^{2,0}$ or $\chi_{0,0,2}^{0,3}$ and $\chi_{2,0,0}^{0,3}$ have the same dimension so only one representative is written

Selection rules for $\chi_\nu^{\lambda,\mu}$ are simply: $\nu_{\frac{N-1}{2}} = 0 \bmod 2$. Field identification occurs due to SO(N) automorphism and results in the rule: $\nu_0 \geq \nu_1$. The comarks of SO(N) for odd N are: (1,2,...,2,1). Representations with their conformal weights are given in the table:

$h_{(0,\ldots,0)} = 0$	$h_{(\underbrace{0,\ldots,1}_{m<\frac{N-1}{2}},\ldots,0)} = \frac{m(N-m)}{2(N+2)}$	$h_{(\underbrace{\underbrace{0,\ldots,0,1}_{m<\frac{N-1}{2}},0\ldots,0,1}_{l<\frac{N-1}{2}},0,\ldots,0)} = \frac{m(N-m)+l(N-l)+2\min[m,l]}{2(N+2)}$
$h_{(0,\ldots,0,2)} = \frac{N^2-1}{8(N+2)}$	$h_{(0,\ldots,0,4)} = \frac{(N-1)(N+3)}{4(N+2)}$	$h_{(\underbrace{0,\ldots,0,1}_{m<\frac{N-1}{2}},0,\ldots,0,2)} = \frac{\frac{N^2-1}{4}+m(N+2-m)}{2(N+2)}$

Fractional part of conformal dimension of the field $\chi_\nu^{\lambda,\mu}$ is calculated as[7]:

$$h_\lambda^{SU(N)_1} + h_\mu^{SU(N)_1} - h_\nu^{SO(N)_4}$$

As in the case of low N the spectrum of q=2 Z_N parafermions is included in the one of the coset and coset fusion rules are just extension of parafermionic ones.

2.7 $\frac{SU(N)_1 \oplus SU(N)_1}{SO(N)_4}$, $N \geq 8$, even

The projection matrix for each of the SU(N) factors is

$$P = \begin{pmatrix} 1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & 1 \\ 0 & 1 & 0 & \ldots & \ldots & \ldots & 0 & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & 2 & 1 & 0 & \ldots & 0 \end{pmatrix} \tag{19}$$

Selection rules for $\chi_\nu^{\lambda,\mu}$ depend on N:

- **N=4k**
  $\lambda_1 + \lambda_{N-1} + \mu_1 + \mu_{N-1} - \nu_1 = \lambda_3 + \lambda_{N-3} + \mu_3 + \mu_{N-3} - \nu_3 = \ldots = \lambda_{\frac{N}{2}-1} + \lambda_{\frac{N}{2}+1} + \mu_{\frac{N}{2}-1} + \mu_{\frac{N}{2}+1} - \nu_{\frac{N}{2}-1} \bmod 2$
  $\nu_{\frac{N}{2}-1} = \nu_{\frac{N}{2}} \bmod 2$

- **N=4k+2**
  $\lambda_1 + \lambda_{N-1} + \mu_1 + \mu_{N-1} - \nu_1 = \lambda_3 + \lambda_{N-3} + \mu_3 + \mu_{N-3} - \nu_3 = \ldots =$

[7]Conformal dimensions of $SU(N)_1$ representations: $h_{\lambda_m} = \frac{m(N-m)}{2N}$.

$\lambda_{\frac{N}{2}-2} + \lambda_{\frac{N}{2}+2} + \mu_{\frac{N}{2}-2} + \mu_{\frac{N}{2}+2} - \nu_{\frac{N}{2}-2}$ mod2
$2\lambda_{\frac{N}{2}} + 2\mu_{\frac{N}{2}} - \nu_{\frac{N}{2}} = -\nu_{\frac{N}{2}-1}$ mod4

Field identifications occur due to SO(N) automorphisms: $A\hat{\omega}_0 = \hat{\omega}_1,\ A\hat{\omega}_0 = \hat{\omega}_{\frac{N}{2}}$ and the even-N SU(N) automorphism: $A\hat{\omega}_0 = \hat{\omega}_{\frac{N}{2}},\ \ (\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) \rightarrow (\lambda_{N-1}, \lambda_{N-2}, \ldots, \lambda_1, \lambda_0)$. It results in the restriction: $\nu_0 \geq$max$(\nu_1, \nu_{\frac{N}{2}-1}, \nu_{\frac{N}{2}})$. The comarks of SO(N) for even N are: (1,2,. . . ,2,1,1). Representations with their conformal weights are given in the table:

$h_{(0,\ldots,0)} = 0$	$h_{(\underbrace{0,\ldots,1}_{m\leq\frac{N}{2}-2},\ldots,0)} = \frac{m(N-1)}{2(N+2)}$	$h_{(\underbrace{\underbrace{0,\ldots,0,1}_{m\leq\frac{N}{2}-2},0\ldots,0,1}_{l\leq\frac{N}{2}-2},0,\ldots,0)} = \frac{(m+l)(N-1)+2\min(m,l)}{2(N+2)}$
$h_{(2,0,\ldots,0)} = \frac{N}{N+2}$	$h_{(0,\ldots,0,1,1)} = \frac{N-2}{8}$	$h_{(0,\ldots,0,2)} = h_{(0,\ldots,2,0)} = \frac{N^2}{8(N+2)}$

As in the case of even N the spectrum of q=2 Z_N extended parafermions is included in the one of the coset and coset fusion rules are just extension of parafermionic ones.

3 Generalized pseudoaffine theory

In the previous section we obtained q=2 parafermions with fusions similar to the ones needed to build the pseudoaffine theory. To obtain extended algebra similar to the pseudoaffine one we should multiply the parafermionic theory with the U(1) boson at suitable level. For N-odd, the level is 2N, while for N-even, the N/2 level should be taken. Consider this in details:

The odd-N case

In the spirit of decomposition (6) first the products $C^0_\lambda \theta_\lambda$ are considered, since they form the extended algebra. As is seen from the spectrum table, the fields of N=3 coset that have the same conformal dimensions as q-parafermions (C^0_λ) are $\chi^{0,1}_0, \chi^{0,2}_0, \chi^{1,0}_0, \chi^{2,0}_0$. In fact for general N the fields that correspond to C^0_λ belong to the set: $\chi^{0,n}_0, \chi^{n,0}_0$. In particular the fields that have the same conformal dimension as the "generating q parafermion" (ψ_1) are : $\chi^{0,2}_0, \chi^{0,N-2}_0, \chi^{2,0}_0, \chi^{N-2,0}_0$.

Let's choose $\chi^{0,2}_0$ as ψ_1, then by taking products with itself one has[8] $\psi_n \sim [\chi^{0,2}_0]^n = \chi^{0,(2n\mathrm{mod}N)}_0$. Among the possible representations of U(1) at

[8]The fusions of $SU(N)_1$ are just: $[\lambda_m] \times [\lambda_n] = [\lambda_{(m+n)\mathrm{modN}}]$

level 2N one seeks for those that complete the dimensions of $\chi_0^{0,2}$ to integer. This condition: $\frac{2(N-1)}{N} + \frac{m^2}{4*2N} \in Z,\ m \in [-2N+1, 2N]$ has only one root: $m = 4$. So the extended algebra contains fields: $\chi_0^{0,2n}\theta_{4n}^{(2N)}$.

The next step is to combine different representations of the product into irreducible representations of extended algebra and to discard representations that are nonlocal with respect to the extended algebra. It is sufficient to check the fusions with the generating operator: $\chi_0^{0,2}\theta_4^{(2N)}$:

$$[\chi_0^{0,2}\theta_4^{(2N)}] \times [\chi_\nu^{\lambda,\mu}\theta_m^{(2N)}] = [\chi_\nu^{\lambda,(\mu+2)}\theta_{(m+4)}^{(2N)}] \tag{20}$$

So the condition of locality is: $m = 2\mu \mathrm{mod N}$.
Consider now the spectrum of the pseudoaffine algebra. Since N is odd any representation of the extended algebra can be denoted by the fields of the form: $\chi_\nu^{m,0}\theta_{n*N}^{(2N)}, n = 0, \pm 1, 2$. Here m is SU(N) representation with $h_m = \frac{m(N-m)}{N}$. $\theta_{n*N}^{(2N)}$ has conformal weights: $0, \frac{1}{8N}, \frac{1}{2N}$ for $n = 0, \pm 1, 2$ respectively. The ν is a representation of $SO(N)_4$ allowed by selection rules and field identifications. From this data the conformal dimension of any representation of extended algebra can be obtained. Finally, the characters of the extended algebra are:

$$\sum_m \chi_\nu^{\lambda,2m}\theta_{4m+n*N}^{(2N)}, n = 0, \pm 1, 2 \tag{21}$$

It is instructive to calculate the number of primaries of the extended algebra: since λ in (21) gets $N+1$ values this number is: $(N+1) \times 4 \times$(number of "possible ν's"). The later is $\frac{(N+1)(N+3)}{8}$ for $N \geq 7$ odd with the only exception for $N = 5$ where it is 4. So the number of fields is an integer multiple of that of the affine case. One can define the "index"$= \frac{\text{Number of fields in the spectrum of pseudoaffine theory}}{\text{Number of fields in the spectrum of affine theory}}$ For N odd the index is $\frac{(N+1)(N+3)}{2}$ and 16 for $N = 5$.

The even-N case

.

As in the case of odd N, the "generating" parafermion is: $\chi_0^{0,2}$, and the boson that completes its dimension to integer is $\theta_2^{(\frac{N}{2})}$. The check of representations local to this algebra gives: $\chi_\nu^{\lambda\mu}\theta_m^{(\frac{N}{2})}$ is local provided: $\mu = m \text{ mod} N$, that in our case ($m \in -\frac{N}{2}+1 \ldots \frac{N}{2}$, $\mu = 0 \ldots N-1$) means just $\mu = m$.

The spectrum of the extended algebra is calculated similarly to the odd-N case, the only difference is in the $SO(N)$ part. Finally the characters are:

$$\sum_m \chi_\nu^{\lambda(\mu+2m)} \theta_{\mu+2m}^{(\frac{N}{2})} \tag{22}$$

Similarly to the N odd case the number of fields in the spectrum: $(N+1)\left(\frac{(\frac{N}{2})^2-\frac{N}{2}}{2}+4\right)$ for $N \geq 8$, even. The index is therefore: $\frac{(\frac{N}{2})^2-\frac{N}{2}}{2}+4$

4 Spectra for small N

The following are spectra of extended algebra for several low N.

N=3

$\chi_0^{0,0} * \theta_0^{(6)}$	0	$\chi_2^{0,0} * \theta_0^{(6)}$	$\frac{4}{5}$	$\chi_4^{0,0} * \theta_0^{(6)}$	$\frac{2}{5}$
$\chi_0^{1,0} * \theta_0^{(6)}$	$\frac{4}{3}$	$\chi_2^{1,0} * \theta_0^{(6)}$	$\frac{2}{15}$	$\chi_4^{1,0} * \theta_0^{(6)}$	$\frac{11}{15}$
$\chi_0^{2,0} * \theta_0^{(6)}$	$\frac{4}{3}$	$\chi_2^{2,0} * \theta_0^{(6)}$	$\frac{2}{15}$	$\chi_4^{2,0} * \theta_0^{(6)}$	$\frac{11}{15}$
$\chi_0^{0,0} * \theta_{\pm 3}^{(6)}$	$\frac{3}{8}$	$\chi_2^{0,0} * \theta_{\pm 3}^{(6)}$	$\frac{47}{40}$	$\chi_4^{0,0} * \theta_{\pm 3}^{(6)}$	$\frac{31}{40}$
$\chi_0^{1,0} * \theta_{\pm 3}^{(6)}$	$\frac{41}{24}$	$\chi_2^{1,0} * \theta_{\pm 3}^{(6)}$	$\frac{61}{120}$	$\chi_4^{1,0} * \theta_{\pm 3}^{(6)}$	$\frac{133}{120}$
$\chi_0^{2,0} * \theta_{\pm 3}^{(6)}$	$\frac{41}{24}$	$\chi_2^{2,0} * \theta_{\pm 3}^{(6)}$	$\frac{61}{120}$	$\chi_4^{2,0} * \theta_{\pm 3}^{(6)}$	$\frac{133}{120}$
$\chi_0^{0,0} * \theta_6^{(6)}$	$\frac{3}{2}$	$\chi_2^{0,0} * \theta_6^{(6)}$	$\frac{23}{10}$	$\chi_4^{0,0} * \theta_6^{(6)}$	$\frac{19}{10}$
$\chi_0^{1,0} * \theta_6^{(6)}$	$\frac{17}{6}$	$\chi_2^{1,0} * \theta_6^{(6)}$	$\frac{49}{30}$	$\chi_4^{1,0} * \theta_6^{(6)}$	$\frac{67}{30}$
$\chi_0^{2,0} * \theta_6^{(6)}$	$\frac{17}{6}$	$\chi_2^{2,0} * \theta_6^{(6)}$	$\frac{49}{30}$	$\chi_4^{2,0} * \theta_6^{(6)}$	$\frac{67}{30}$

N=4

$\chi_0^{0,0} * \theta_0^{(2)}$	0	$\chi_{0,2}^{0,0} * \theta_0^{(2)}$	$\frac{2}{3}$	$\chi_{2,0}^{0,0} * \theta_0^{(2)}$	$\frac{2}{3}$	$\chi_{2,2}^{0,0} * \theta_0^{(2)}$	$\frac{1}{3}$
$\chi_{1,1}^{1,0} * \theta_0^{(2)}$	$\frac{1}{8}$	$\chi_{1,1}^{0,1} * \theta_1^{(2)}$	$\frac{1}{4}$	$\chi_{1,1}^{2,1} * \theta_1^{(2)}$	$\frac{3}{4}$		
$\chi_0^{1,1} * \theta_0^{(2)}$	$\frac{7}{8}$	$\chi_{0,2}^{1,1} * \theta_0^{(2)}$	$\frac{13}{24}$	$\chi_{2,0}^{1,1} * \theta_1^{(2)}$	$\frac{13}{24}$	$\chi_{2,2}^{1,1} * \theta_1^{(2)}$	$\frac{5}{24}$
$\chi_0^{2,0} * \theta_0^{(2)}$	$\frac{1}{2}$	$\chi_{0,2}^{2,0} * \theta_0^{(2)}$	$\frac{1}{6}$	$\chi_{2,0}^{2,0} * \theta_0^{(2)}$	$\frac{1}{6}$	$\chi_{2,2}^{2,0} * \theta_0^{(2)}$	$\frac{5}{6}$
$\chi_{1,1}^{3,0} * \theta_0^{(2)}$	$\frac{1}{8}$						
$\chi_0^{3,1} * \theta_0^{(2)}$	$\frac{7}{8}$	$\chi_{0,2}^{3,1} * \theta_0^{(2)}$	$\frac{13}{24}$	$\chi_{2,0}^{3,1} * \theta_1^{(2)}$	$\frac{13}{24}$	$\chi_{2,2}^{3,1} * \theta_1^{(2)}$	$\frac{5}{24}$

To save the space, for $N = 5$ part of the spectrum is omitted, e.g. the fields $\chi_\nu^{l,0}$ and $\chi_\nu^{(N-l),0}$ have the same dimensions and similar fusions.

N=5

$\chi_{(0,0)}^{0,0} * \theta_0^{(10)}$	0	$\chi_{(1,0)}^{0,0} * \theta_0^{(10)}$	$\frac{5}{7}$	$\chi_{(0,2)}^{0,0} * \theta_0^{(10)}$	$\frac{4}{7}$	$\chi_{(2,0)}^{0,0} * \theta_0^{(10)}$	$\frac{2}{7}$	$\chi_{(1,2)}^{0,0} * \theta_0^{(10)}$	$\frac{1}{7}$	$\chi_{(0,4)}^{0,0} * \theta_0^{(10)}$	$\frac{6}{7}$
$\chi_{(0,0)}^{1,0} * \theta_0^{(10)}$	$\frac{2}{5}$	$\chi_{(1,0)}^{1,0} * \theta_0^{(10)}$	$\frac{4}{35}$	$\chi_{(0,2)}^{1,0} * \theta_0^{(10)}$	$\frac{34}{35}$	$\chi_{(2,0)}^{1,0} * \theta_0^{(10)}$	$\frac{24}{35}$	$\chi_{(1,2)}^{1,0} * \theta_0^{(10)}$	$\frac{19}{35}$	$\chi_{(0,4)}^{1,0} * \theta_0^{(10)}$	$\frac{9}{35}$
$\chi_{(0,0)}^{2,0} * \theta_0^{(10)}$	$\frac{3}{5}$	$\chi_{(1,0)}^{2,0} * \theta_0^{(10)}$	$\frac{11}{35}$	$\chi_{(0,2)}^{2,0} * \theta_0^{(10)}$	$\frac{6}{35}$	$\chi_{(2,0)}^{2,0} * \theta_0^{(10)}$	$\frac{31}{35}$	$\chi_{(1,2)}^{2,0} * \theta_0^{(10)}$	$\frac{26}{35}$	$\chi_{(0,4)}^{2,0} * \theta_0^{(10)}$	$\frac{16}{35}$
$\chi_{(0,0)}^{0,0} * \theta_{\pm 5}^{(10)}$	$\frac{5}{8}$	$\chi_{(1,0)}^{0,0} * \theta_{\pm 5}^{(10)}$	$\frac{75}{56}$	$\chi_{(0,2)}^{0,0} * \theta_{\pm 5}^{(10)}$	$\frac{67}{56}$	$\chi_{(2,0)}^{0,0} * \theta_{\pm 5}^{(10)}$	$\frac{51}{56}$	$\chi_{(1,2)}^{0,0} * \theta_{\pm 5}^{(10)}$	$\frac{1}{7}$	$\chi_{(0,4)}^{0,0} * \theta_{\pm 5}^{(10)}$	$\frac{83}{56}$
$\chi_{(0,0)}^{1,0} * \theta_{\pm 5}^{(10)}$	$\frac{41}{40}$	$\chi_{(1,0)}^{1,0} * \theta_{\pm 5}^{(10)}$	$\frac{207}{280}$	$\chi_{(0,2)}^{1,0} * \theta_{\pm 5}^{(10)}$	$\frac{167}{280}$	$\chi_{(2,0)}^{1,0} * \theta_{\pm 5}^{(10)}$	$\frac{367}{280}$	$\chi_{(1,2)}^{1,0} * \theta_{\pm 5}^{(10)}$	$\frac{327}{280}$	$\chi_{(0,4)}^{1,0} * \theta_{\pm 5}^{(10)}$	$\frac{249}{280}$
$\chi_{(0,0)}^{2,0} * \theta_{\pm 5}^{(10)}$	$\frac{49}{40}$	$\chi_{(1,0)}^{2,0} * \theta_{\pm 5}^{(10)}$	$\frac{263}{280}$	$\chi_{(0,2)}^{2,0} * \theta_{\pm 5}^{(10)}$	$\frac{223}{280}$	$\chi_{(2,0)}^{2,0} * \theta_{\pm 5}^{(10)}$	$\frac{143}{280}$	$\chi_{(1,2)}^{2,0} * \theta_{\pm 5}^{(10)}$	$\frac{383}{280}$	$\chi_{(0,4)}^{2,0} * \theta_{\pm 5}^{(10)}$	$\frac{303}{280}$
$\chi_{(0,0)}^{0,0} * \theta_{10}^{(10)}$	$\frac{5}{2}$	$\chi_{(1,0)}^{0,0} * \theta_{10}^{(10)}$	$\frac{45}{14}$	$\chi_{(0,2)}^{0,0} * \theta_{10}^{(10)}$	$\frac{43}{14}$	$\chi_{(2,0)}^{0,0} * \theta_{10}^{(10)}$	$\frac{39}{14}$	$\chi_{(1,2)}^{0,0} * \theta_{10}^{(10)}$	$\frac{37}{14}$	$\chi_{(0,4)}^{0,0} * \theta_{10}^{(10)}$	$\frac{47}{14}$
$\chi_{(0,0)}^{1,0} * \theta_{10}^{(10)}$	$\frac{29}{10}$	$\chi_{(1,0)}^{1,0} * \theta_{10}^{(10)}$	$\frac{183}{70}$	$\chi_{(0,2)}^{1,0} * \theta_{10}^{(10)}$	$\frac{173}{70}$	$\chi_{(2,0)}^{1,0} * \theta_{10}^{(10)}$	$\frac{223}{70}$	$\chi_{(1,2)}^{1,0} * \theta_{10}^{(10)}$	$\frac{213}{70}$	$\chi_{(0,4)}^{1,0} * \theta_{10}^{(10)}$	$\frac{193}{70}$
$\chi_{(0,0)}^{2,0} * \theta_{10}^{(10)}$	$\frac{31}{10}$	$\chi_{(1,0)}^{2,0} * \theta_{10}^{(10)}$	$\frac{197}{70}$	$\chi_{(0,2)}^{2,0} * \theta_{10}^{(10)}$	$\frac{187}{70}$	$\chi_{(2,0)}^{2,0} * \theta_{10}^{(10)}$	$\frac{237}{70}$	$\chi_{(1,2)}^{2,0} * \theta_{10}^{(10)}$	$\frac{227}{70}$	$\chi_{(0,4)}^{2,0} * \theta_{10}^{(10)}$	$\frac{207}{70}$

5 Conclusions

In this paper generalized pseudoaffine theories are defined. They come instead of pseudoaffine theories when forbidden q is considered. These theories are obtained as a product of generalized parafermions with the generalized bosons. A specific example of $q = 2$ Z_N parafermions is considered. The generalized pseudoaffine theory that is obtained has several remarkable properties: number of fields in the spectrum is an integer multiple of that for $SU(2)_N$, for every field ϕ_i of $SU(2)_N$ there is a field $\hat{\phi}_i$ with dimension $q\Delta(\phi_i)$ and fusion coefficient: $N_{ij}^k = \hat{N}_{ij}^k$. The central charge of the generalized pseudoaffine theory is $qc_{SU(2)_N} \mathrm{mod}\ 4$. In fact by multiplying the parafermions with different generalized bosons (as explained in the introduction) an infinite hierarchy of such theories is obtained.

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